CSC1016

Welcome (again!)

Get course notes handout

(and *read* them)

The question of whether computers can think is like the question of whether submarines can swim.

Edsger Dijkstra

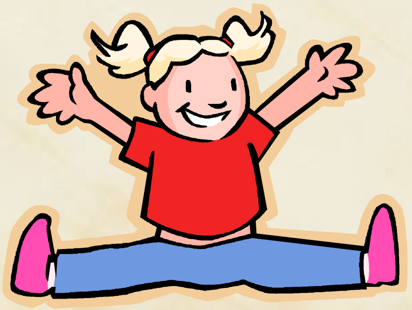
Me - for those who don’t know

- Mike Linck
  - Pam.Linck@ebucksmail.com

- Room

You - refresher

- what I expect:
  - Read the book chapter before class
  - Participate in class
  - Do the Self-test exercises!
  - Let me know of problems/issues with you have with the work
- What I'd like:
  - Enthusiasm
  - Suggestions



Chapter 11

Recursion

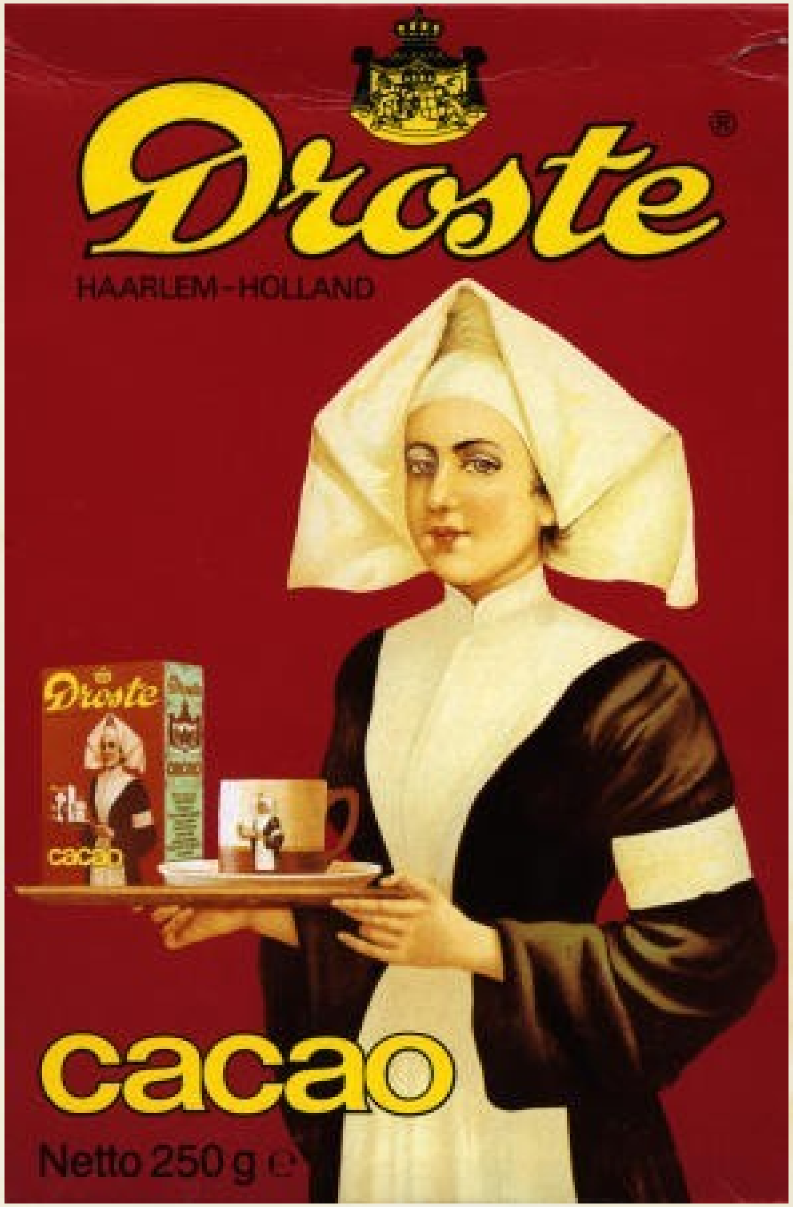
Recursion

- *It was a dark and stormy night, and the head of the brigands said to Antonio:"Antonio, tell us a tale". And so Antonio began:*
  - "It was a dark and stormy night and the head of the brigands said to Antonio, "Antonio, tell us a tale". And so Antonio began:
    - "It was a dark and stormy night….

From wikipedia:

- a recursive definition of person's ancestors:
  - One's parents are one's ancestors (**base case**);
  - The parents of any ancestor are also ancestors of the person under consideration (**recursion step**).

Also from Wikipedia:



- A visual form of recursion known as the Droste effect.

Recursion

- Classic introductory example - Factorial function:

$$n! = 1 \times \ldots \times (n-1) \times n$$

$n! = (n-1)! \times n$ for n>=1

$$0! = 1$$

Recursion - another example

- Fibonacci sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

–recursive sequence**,**

–calculate the next term by summing the preceding two

$$f(n) = f(n-1) + f(n-2)$$

$$f(1) = 1$$

$$f(2) = 1$$

Recursive `void` Methods

- A *recursive* method is a method that includes a call to itself
  - based on the general problem solving technique of breaking down a task into subtasks
  - In particular, recursion can be used whenever **one subtask is a smaller version of the original task**

Vertical Numbers

- The *static recursive method* from Ch11:
  - **`writeVertical:`**
    - takes one (nonnegative) **`int`** argument
    - writes the digits of that **`int`** down the screen, one per line
      - Note: Recursive methods need not be static
    - e.g:
      - input:

        10051

      - output:

        1
        0
        0
        5
        1

Vertical Numbers

- This task may be broken down into the following two subtasks:
  - Simple case:
    - If n<10, then write the number n to the screen
  - Recursive Case:
    - If n>=10, then do two subtasks:
      - Output all the digits except the last digit
      - Output the last digit

Vertical Numbers

- Given the argument 1234, the output of the first subtask would be:

  ```
  1
  2
  3
  ```

- The output of the second part would be:

  ```
  4
  ```

Vertical Numbers

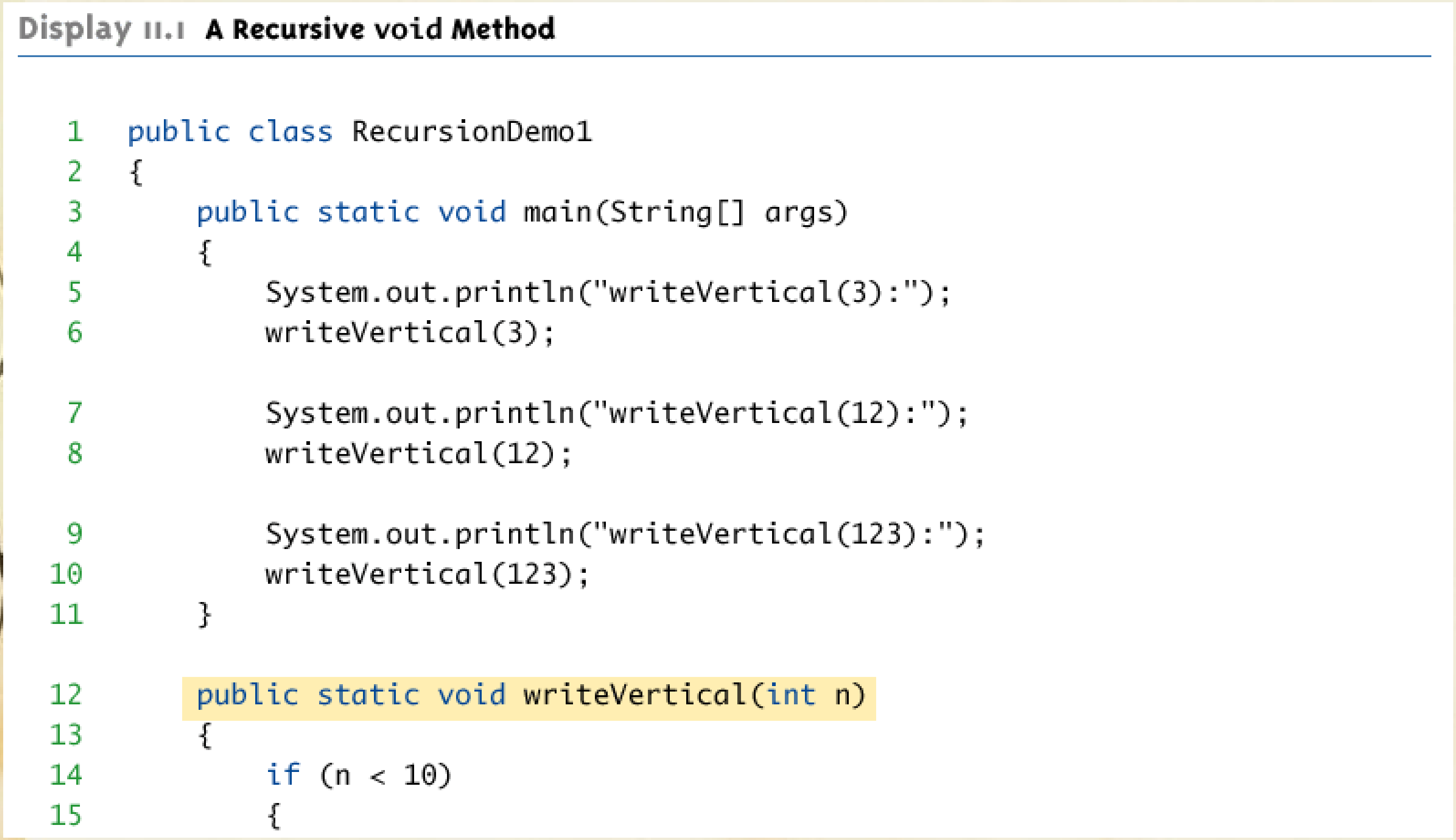
- The decomposition of tasks into subtasks can be used to derive the method definition:
  - Subtask 1 is a smaller version of the original task, so it can be implemented with a recursive call
  - Subtask 2 is just the simple case

Algorithm for Vertical Numbers

- Given parameter **n**:

```
if (n<10)
   System.out.println(n);
else
{
   writeVertical(the number n with the last digit removed);
   System.out.println(the last digit of n);
}
```

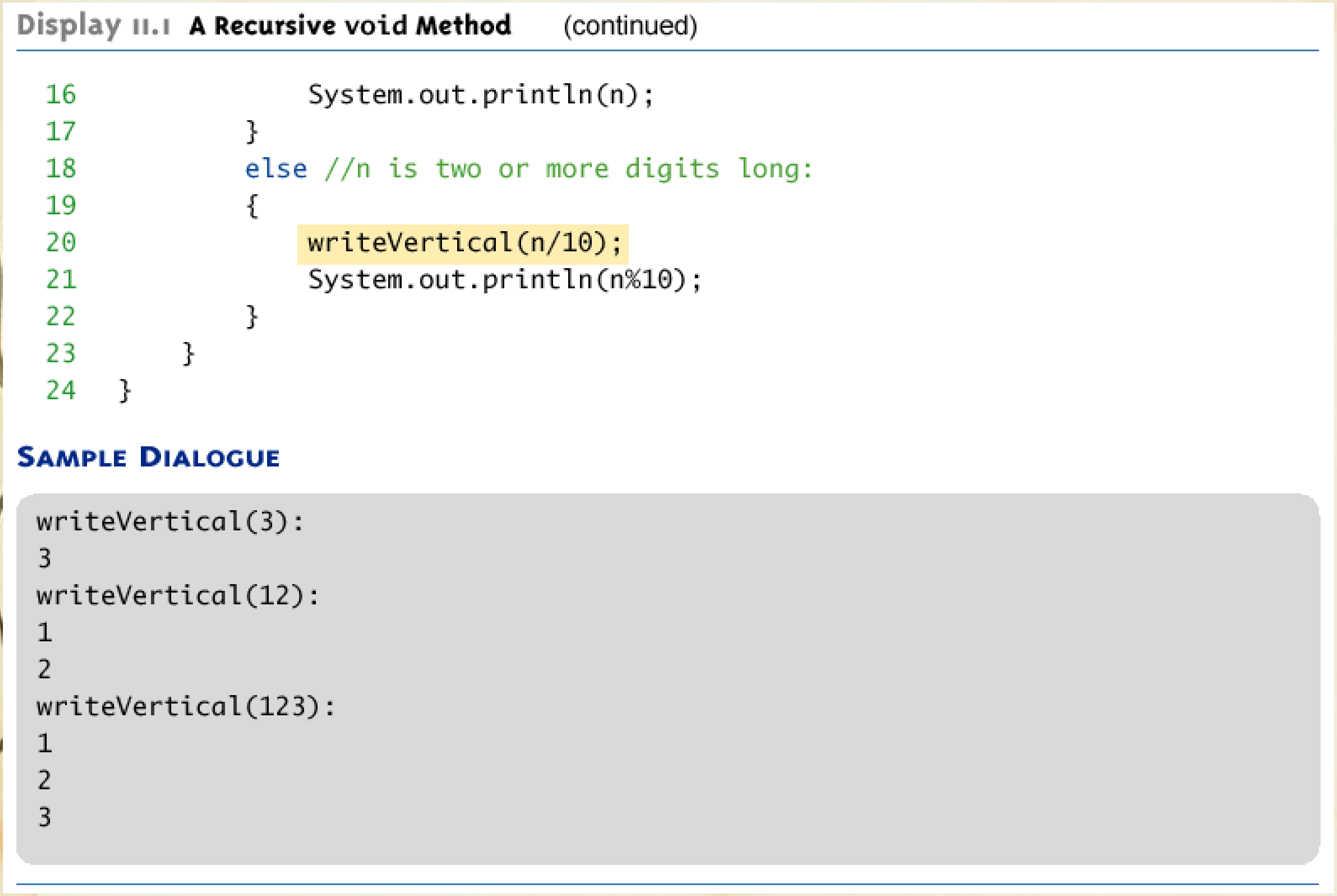
A Recursive **`void`** Method (Part 1 of 2)



Display 11.1 A Recursive void Method

```java
public class RecursionDemo1
{
    public static void main(String[] args)
    {
        System.out.println("writeVertical(3):");
        writeVertical(3);

        System.out.println("writeVertical(12):");
        writeVertical(12);

        System.out.println("writeVertical(123):");
        writeVertical(123);
    }

    public static void writeVertical(int n)
    {
        if (n < 10)
        {
```

A Recursive `void` Method



Display 11.1 **A Recursive void Method** (continued)

```
                System.out.println(n);
            }
            else //n is two or more digits long:
            {
                writeVertical(n/10);
                System.out.println(n%10);
            }
        }
    }
```

Sample Dialogue

```
writeVertical(3):
3
writeVertical(12):
1
2
writeVertical(123):
1
2
3
```

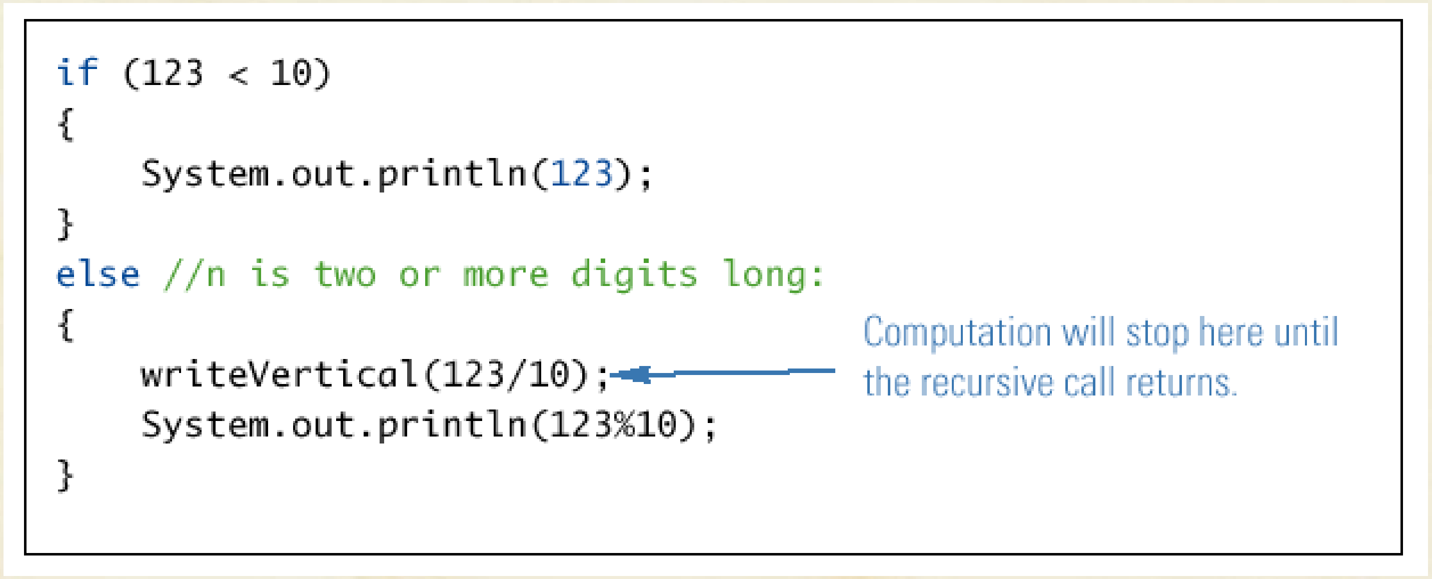
Tracing a Recursive Call

- Recursive methods are processed in the same way as any method call

```
writeVertical(123);
```

  - When this call is executed, the argument `123` is substituted for the parameter `n`, and the body of the method is executed
  - Since `123` is not less than `10`, the `else` part is executed

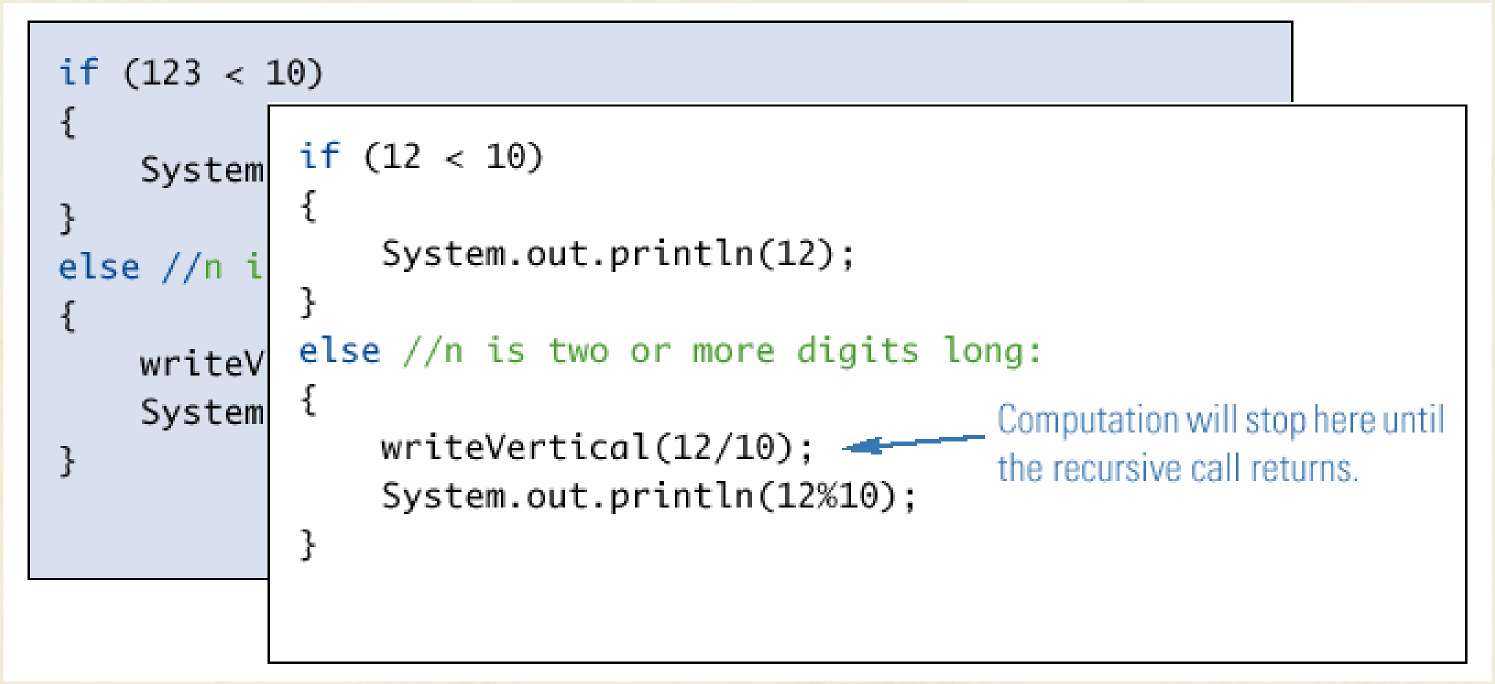
Execution of **`writeVertical(123)`**



```
if (123 < 10)
{
    System.out.println(123);
}
else //n is two or more digits long:
{
    writeVertical(123/10);
    System.out.println(123%10);
}
```

Computation will stop here until the recursive call returns.

Execution of **`writeVertical(12)`**

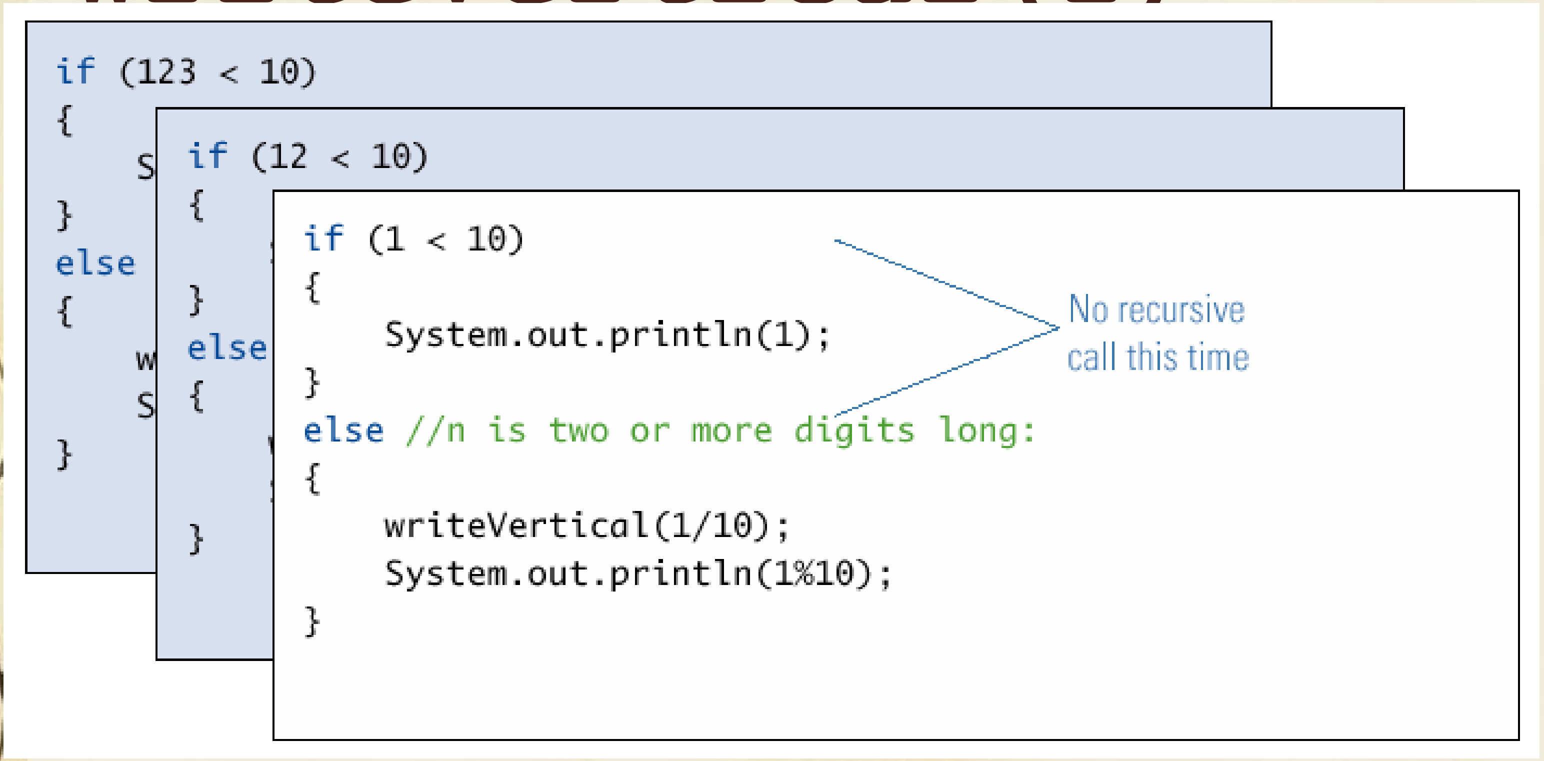


```
if (123 < 10)
{
    System
}
else //n i
{
    writeV
    System
}
```

```
if (12 < 10)
{
    System.out.println(12);
}
else //n is two or more digits long:
{
    writeVertical(12/10);
    System.out.println(12%10);
}
```

Computation will stop here until the recursive call returns.

Execution of **`writeVertical(1)`**

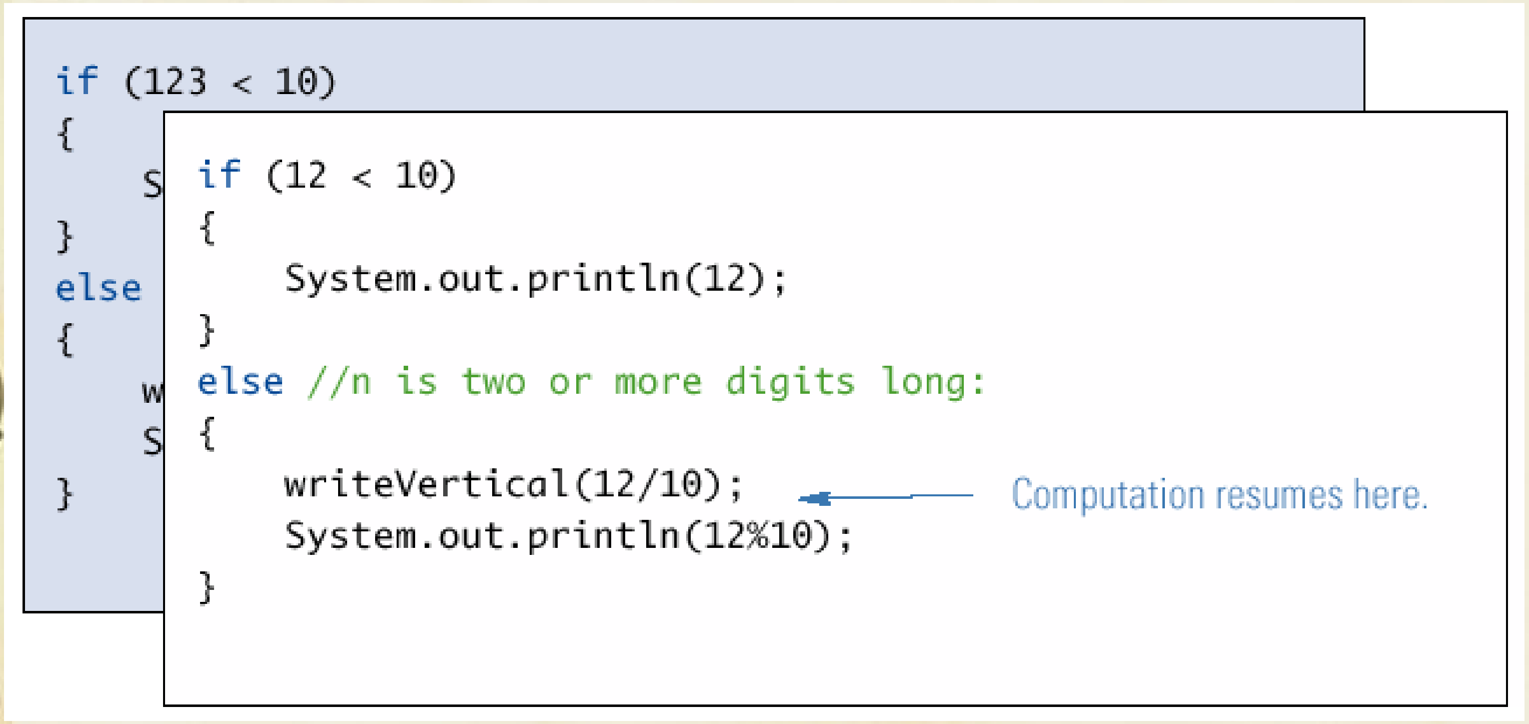


```
if (123 < 10)
{
    S
}
else
{
    w
    S
}
```

```
if (12 < 10)
{

}
else
{

}
```

```
if (1 < 10)
{
    System.out.println(1);
}
else //n is two or more digits long:
{
    writeVertical(1/10);
    System.out.println(1%10);
}
```

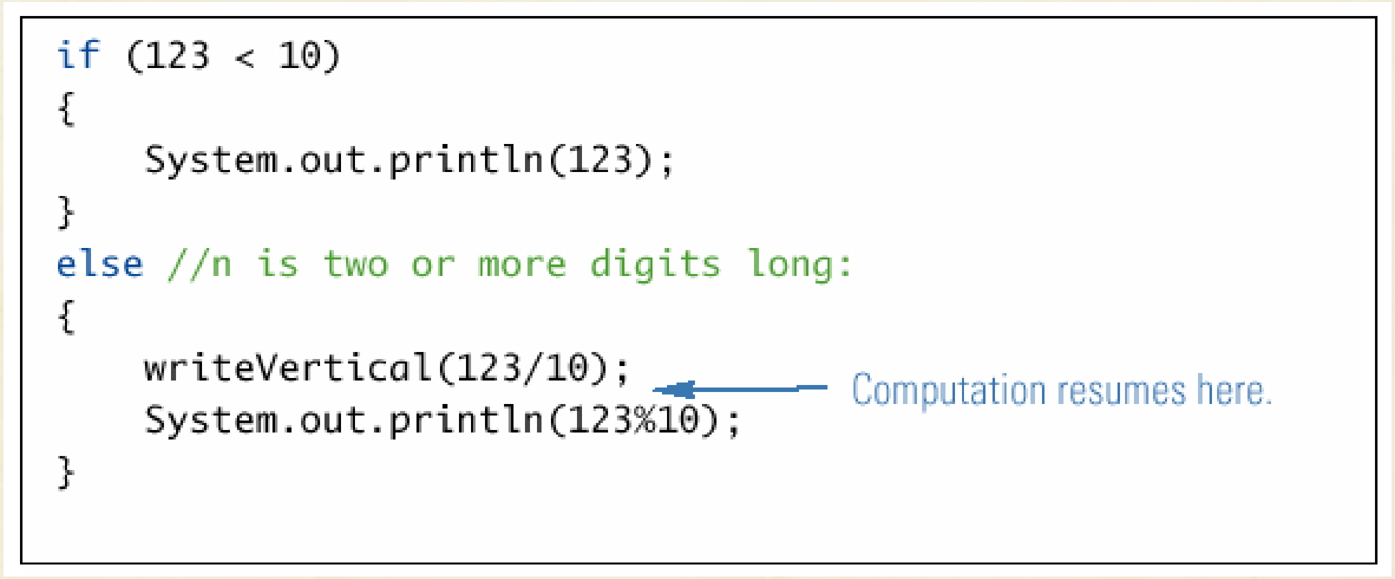
Completion of **`writeVertical(12)`**



```
if (123 < 10)
{
    S
}
else
{
    w
    S
}
```

```
if (12 < 10)
{
    System.out.println(12);
}
else //n is two or more digits long:
{
    writeVertical(12/10);
    System.out.println(12%10);
}
```

Completion of **`writeVertical(123)`**



```java
if (123 < 10)
{
    System.out.println(123);
}
else //n is two or more digits long:
{
    writeVertical(123/10);
    System.out.println(123%10);
}
```

A Closer Look at Recursion

- When the computer encounters a recursive call, it must temporarily suspend its execution of a method
  - It does this because *it must know the result of the recursive call before it can proceed*
  - It saves all the information it needs to continue the computation later on, when it returns from the recursive call
- Ultimately, this entire process terminates when one of the recursive calls does not depend upon recursion to return

General Form of a Recursive Method Definition

- The general outline of a successful recursive method definition is as follows:
  - One or more cases that include **one or more recursive calls** to the method being defined
    - These recursive calls should solve "smaller" versions of the task performed by the method being defined
  - One or more cases that include no recursive

Pitfall: Infinite Recursion

- In the `writeVertical` example, the series of recursive calls eventually reached a call of the method that did not involve recursion (a stopping case)
- If, instead, every recursive call had produced another recursive call, then a call to that method would, in theory, run forever
  - This is called *infinite recursion*
  - In practice, such a method runs until the computer runs out of resources, and the program terminates abnormally

Pitfall: Infinite Recursion

- An alternative version of **writeVertical**
  - Note: No stopping case!

```
public static void
                  newWriteVertical(int n)
{
   newWriteVertical(n/10);
   System.out.println(n%10);
}
```

Pitfall: Infinite Recursion

- A program with this method will compile and run
- Calling `newWriteVertical(12)` causes that execution to stop to execute the recursive call `newWriteVertical(12/10)`
  - Which is equivalent to `newWriteVertical(1)`
- Calling `newWriteVertical(1)` causes that execution to stop to execute the recursive call `newWriteVertical(1/10)`
  - Which is equivalent to `newWriteVertical(0)`

Pitfall: Infinite Recursion

- Calling `newWriteVertical(0)` causes that execution to stop to execute the recursive call `newWriteVertical(0/10)`
  - Which is equivalent to `newWriteVertical(0)`

. . . and so on, forever!

- Since the definition of `newWriteVertical` has no stopping case, the process will proceed *forever* (or until the computer runs out of resources)

Stacks for Recursion

- To keep track of recursion (and other things), most computer systems use a *stack*
  - A stack is a very specialized kind of memory structure analogous to a stack of paper
  - As an analogy, there is also an inexhaustible supply of extra blank sheets of paper
  - Information is placed on the stack by writing on one of these sheets, and placing it on top of the stack (becoming the new top of the stack)
  - More information is placed on the stack by writing on another one of these sheets, placing it on top of the stack, and so on

Stacks for Recursion

  - To get information out of the stack, the top paper can be read, *but only the top paper*
  - To get more information, the top paper can be thrown away, and then the new top paper can be read, and so on
- Since the last sheet put on the stack is the first sheet taken off the stack, a stack is called a *last-in/first-out* memory structure *(LIFO)*

Stacks for Recursion

- To keep track of recursion, whenever a method is called, a new "sheet of paper" is taken
  - The method definition is copied onto this sheet, and the arguments are plugged in for the method parameters
  - The computer starts to execute the method body
  - When it encounters a recursive call, it stops the computation in order to make the recursive call
  - It writes information about the current method on the *sheet of paper*, and places it on the stack

Stacks for Recursion

- A new *sheet of paper* is used for the recursive call
  - The computer writes a second copy of the method, plugs in the arguments, and starts to execute its body
  - When this copy gets to a recursive call, its information is saved on the stack also, and a new *sheet of paper* is used for the new recursive call

Stacks for Recursion

- This process continues until some recursive call to the method completes its computation without producing any more recursive calls
  - Its *sheet of paper* is then discarded
- Then the computer goes to the top *sheet of paper* on the stack
  - This sheet contains the partially completed computation that is waiting for the recursive computation that just ended
  - Now it is possible to proceed with that suspended computation

Stacks for Recursion

- After the suspended computation ends, the computer discards its corresponding sheet of paper (the one on top)
- The suspended computation that is below it on the stack now becomes the computation on top of the stack
- This process continues until the computation on the bottom sheet is completed

Stacks for Recursion

- Depending on how many recursive calls are made, and how the method definition is written, the stack may grow and shrink in any fashion
- The stack of paper analogy has its counterpart in the computer
  - The contents of one of the *sheets of paper* is called a *stack frame* or *activation record*
  - The stack frames don't actually contain a complete copy of the method definition, but reference a single copy instead

Pitfall: Stack Overflow

- There is always some limit to the size of the stack
  - If there is a long chain in which a method makes a call to itself, and that call makes another recursive call, . . . , and so forth, there will be many suspended computations placed on the stack
  - If there are too many, then the stack will attempt to grow beyond its limit, resulting in an error condition known as a *stack overflow*
- A common cause of stack overflow is infinite recursion

Summary so far

- Recursion
  - Recursive void methods
    - Base case (simple case/stopping case)
    - Recursive case
  - Tracing a recursive call
  - Infinite recursion and stack overflow

Recursive versus iterative definition of a line

Line(5) produces:

```
*****
```

Further exercises

- recursive function for drawing a pyramid, of a height n
- e.g. Pyramid(3) displays:

```
  *
 ***
*****
```

  - Suggestions?

Further exercises

- recursive function for drawing a pyramid, of a height n
- e.g. Pyramid(3) displays:

```
  L
 LOL
LOLOL
```

  - Suggestions?

Further exercises

- recursive function for drawing a Christmas tree of pyramids

```
Tree(1,ÓÓ) displays:
 L
LOL

Tree(2,ÓÓ) displays:
  L
 LOL
  L
 LOL
LOLOL
```

```
Tree(3,ÓÓ) displays:
   L
  LOL
   L
  LOL
 LOLOL
   L
  LOL
 LOLOL
LOLOLOL
```

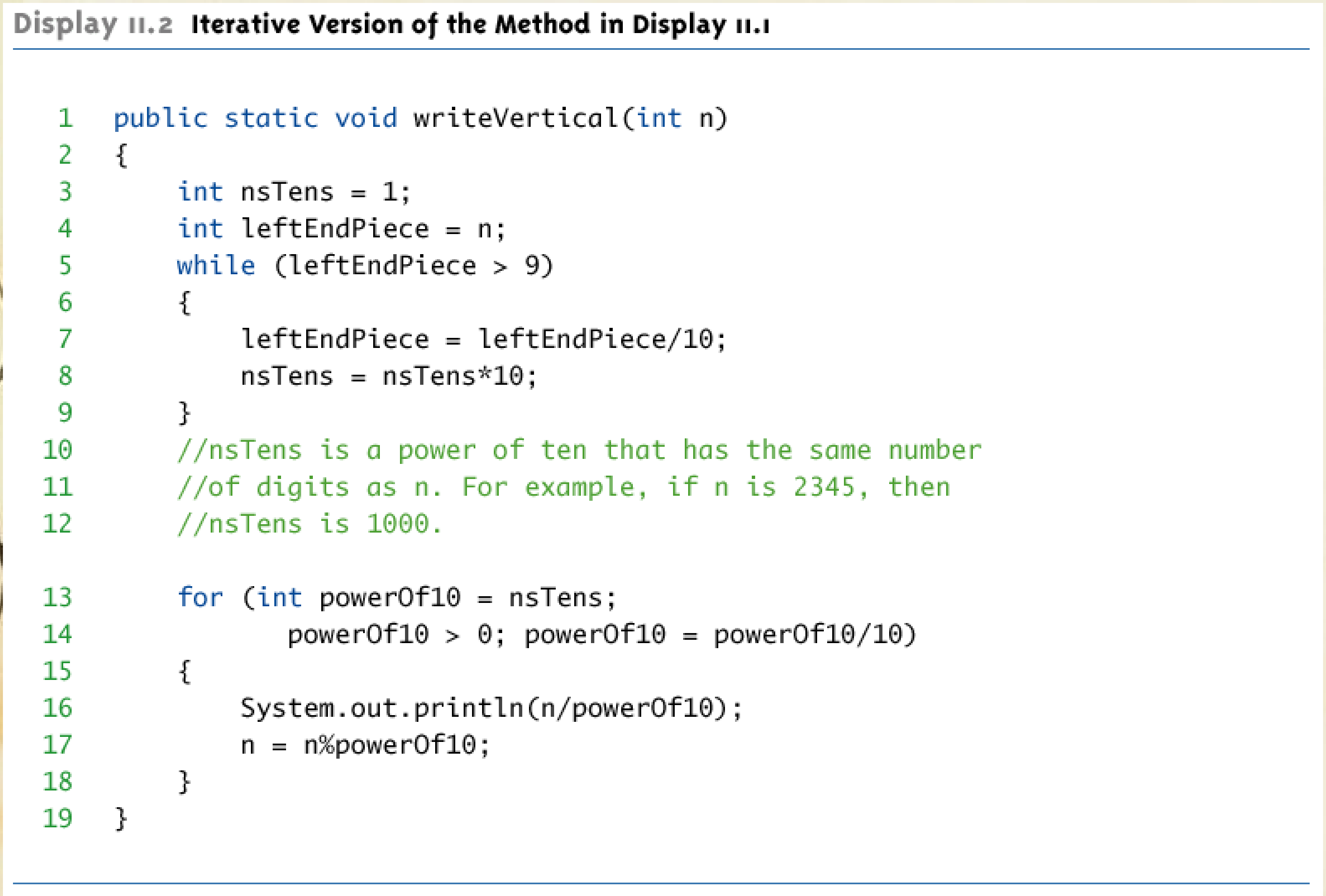
Further exercises

- Towers of Hanoi
  - very old problem
  - tons of examples on the web

Recursion Versus Iteration

- Recursion is not absolutely necessary
  - Any task that can be done using recursion can also be done in a nonrecursive manner
  - A nonrecursive version of a method is called an *iterative version*
- An iteratively written method will typically use loops of some sort in place of recursion
- A recursively written method can be simpler, but will usually run slower and use more storage than an equivalent iterative version

Iterative version of **`writeVertical`**



Display 11.2 Iterative Version of the Method in Display 11.1

```java
public static void writeVertical(int n)
{
    int nsTens = 1;
    int leftEndPiece = n;
    while (leftEndPiece > 9)
    {
        leftEndPiece = leftEndPiece/10;
        nsTens = nsTens*10;
    }
    //nsTens is a power of ten that has the same number
    //of digits as n. For example, if n is 2345, then
    //nsTens is 1000.

    for (int powerOf10 = nsTens;
            powerOf10 > 0; powerOf10 = powerOf10/10)
    {
        System.out.println(n/powerOf10);
        n = n%powerOf10;
    }
}
```

Recursive Methods that Return a Value

- Recursion is not limited to `void` methods
- A recursive method can return a value of any type
- An outline for a successful recursive method that returns a value is as follows:
  - One or more cases in which the value returned is computed in terms of calls to the same method
  - the arguments for the recursive calls should be intuitively "smaller"
  - One or more cases in which the value returned is computed without the use of any recursive calls (**the base or stopping cases**)

Another Powers Method

- The method **`pow`** from the Math class computes powers
  - It takes two arguments of type **`double`** and returns a value of type **`double`**
- The recursive method **`power`** takes two arguments of type **`int`** and returns a value of type **`int`**
  - The definition of **`power`** is based on the following formula:

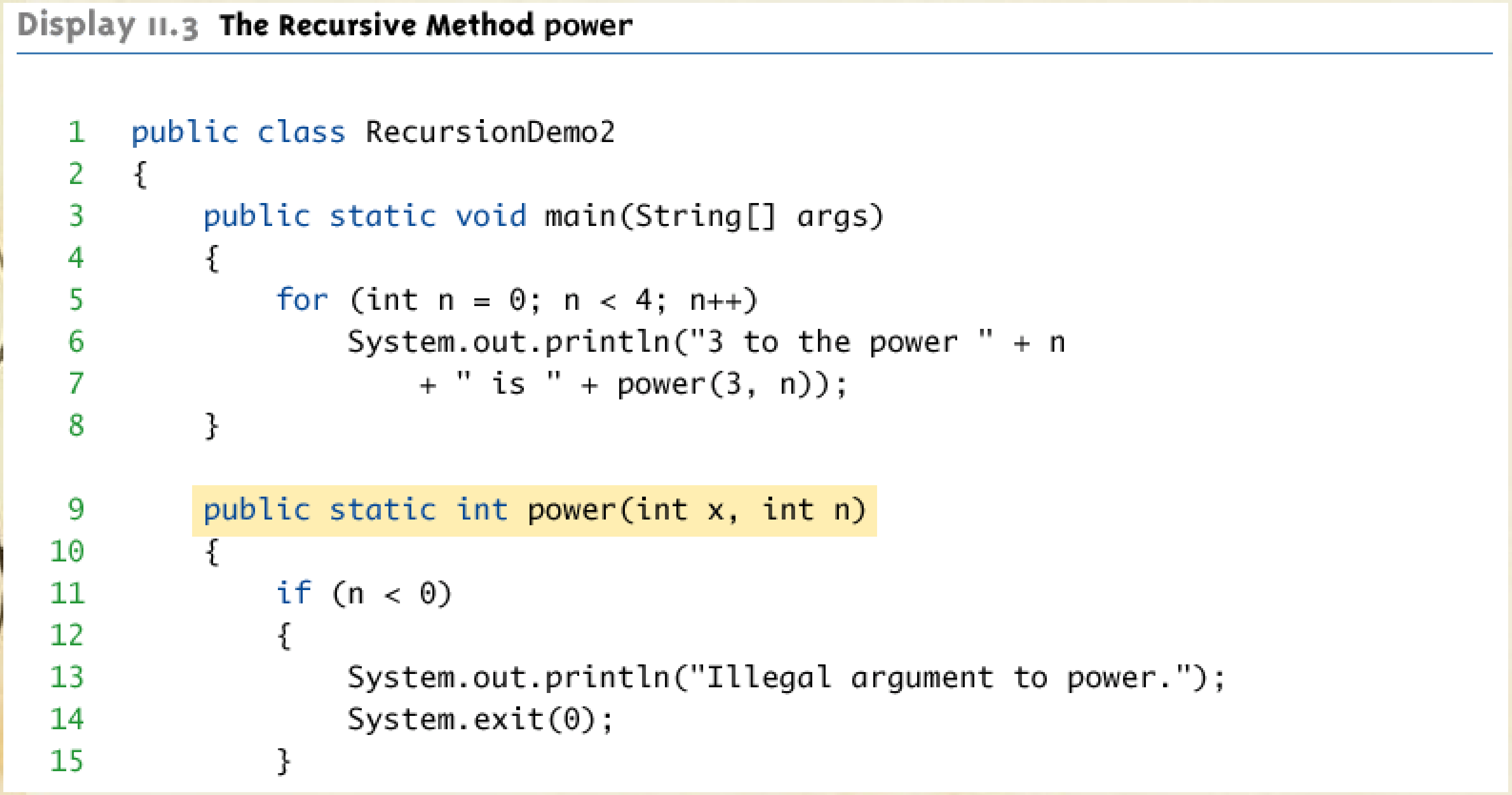
    x^n is equal to $x^{n-1} * x$

Another Powers Method

- In terms of Java, the value returned by `power(x, n)` for `n>0` should be the same as

  `power(x, n-1) * x`
- When `n=0`, then `power(x, n)` should return `1`
  - This is the **stopping case**

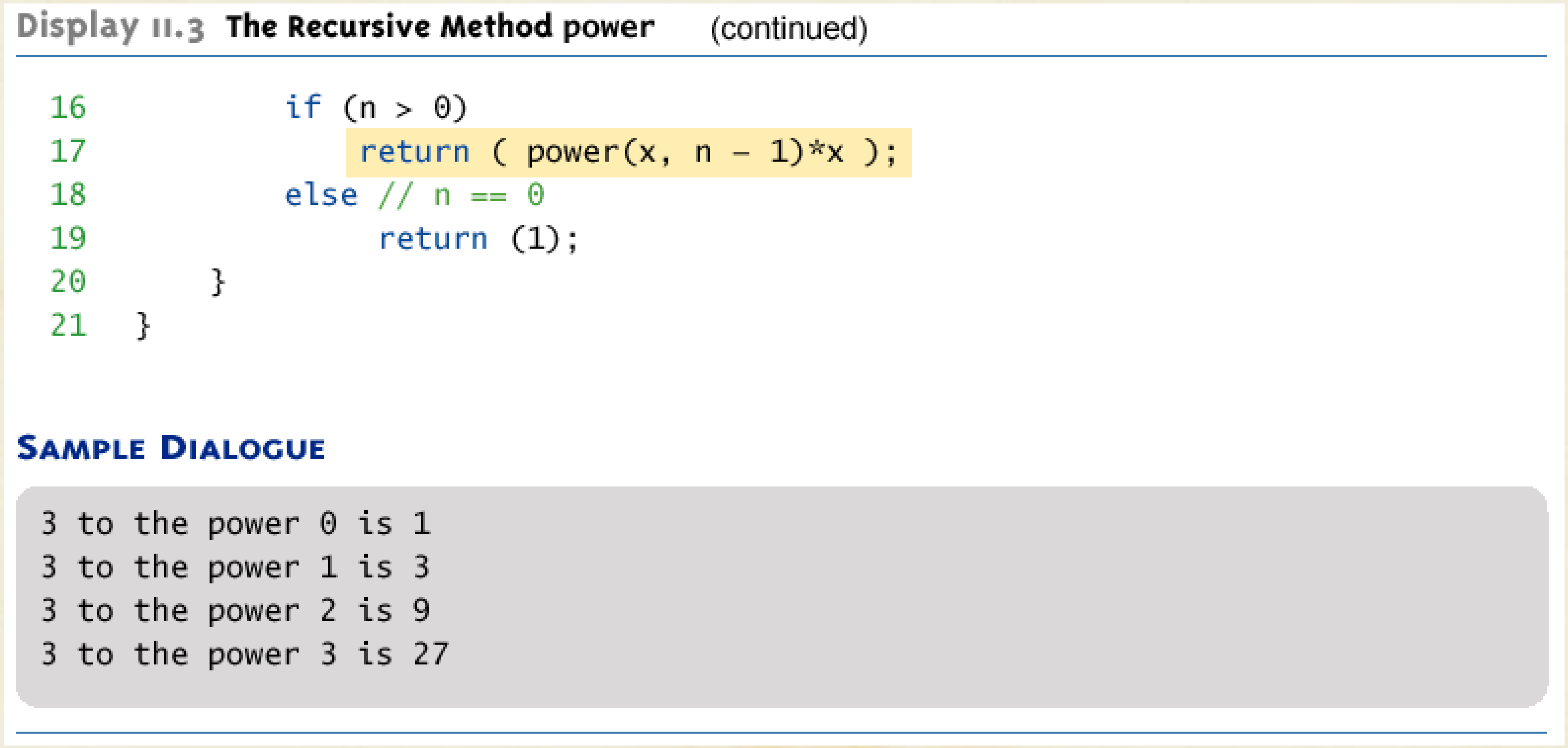
The Recursive Method **`power`** (Part 1 of 2)



Display 11.3 **The Recursive Method power**

```
public class RecursionDemo2
{
    public static void main(String[] args)
    {
        for (int n = 0; n < 4; n++)
            System.out.println("3 to the power " + n
                + " is " + power(3, n));
    }

    public static int power(int x, int n)
    {
        if (n < 0)
        {
            System.out.println("Illegal argument to power.");
            System.exit(0);
        }
```

The Recursive Method `power` (Part 1 of 2)



Display 11.3 The Recursive Method power (continued)

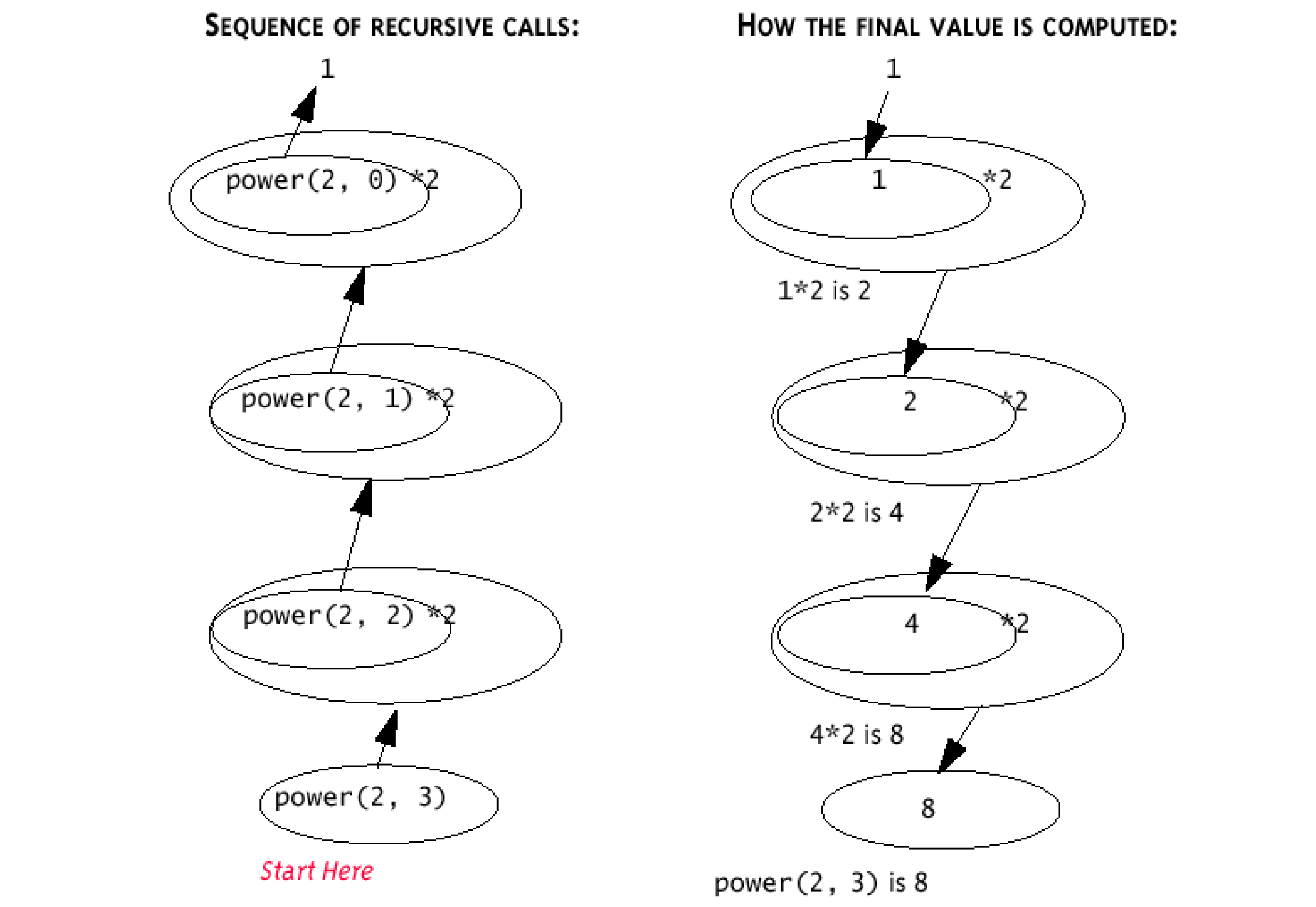
```
        if (n > 0)
            return ( power(x, n - 1)*x );
        else // n == 0
            return (1);
    }
}
```

Sample Dialogue

```
3 to the power 0 is 1
3 to the power 1 is 3
3 to the power 2 is 9
3 to the power 3 is 27
```

Evaluating the Recursive Method Call `power(2,3)`

Display 11.4 **Evaluating the Recursive Method Call power(2,3)**



Thinking Recursively

- If a problem lends itself to recursion, it is more important to think of it in recursive terms, rather than concentrating on the stack and the suspended computations

  `power(x,n)` returns `power(x, n-1) * x`

- In the case of methods that return a value, there are three properties that must be satisfied, as follows:

Thinking Recursively

1. There is no infinite recursion
    - Every chain of recursive calls must reach a stopping case
2. Each stopping case returns the correct value for that case
3. For the cases that involve recursion: *if* all recursive calls return the correct value, *then* the final value returned by the method is the correct value

- These properties follow a technique also known as *mathematical induction*

Recursive Design Techniques

- The same rules can be applied to a recursive `void` method:

1. There is no infinite recursion
2. Each stopping case performs the correct action for that case
3. For each of the cases that involve recursion: if all recursive calls perform their actions correctly, then the entire case performs correctly

Binary Search

- Binary search uses a recursive method to search an array to find a specified value
- The array must be a **sorted array**:

  `a[0]≤a[1]≤a[2]≤. . . ≤  a[finalIndex]`

- If the value is found, its index is returned
- If the value is not found, -1 is returned

- Note: Each execution of the recursive

Binary Search

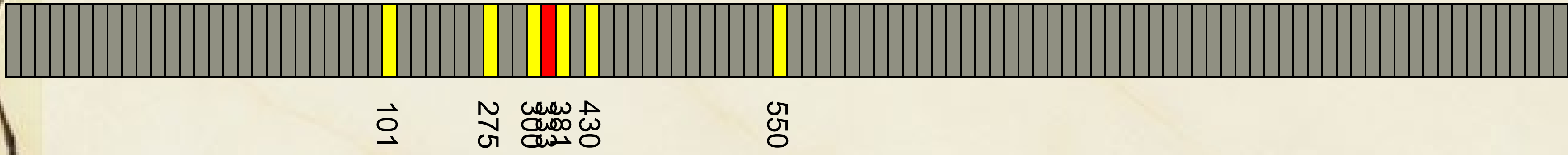
- An algorithm to solve this task looks at the middle of the array or array segment first
- If the value looked for is smaller than the value in the middle of the array
  - Then the second half of the array or array segment can be ignored
  - This strategy is then applied to the first half of the array or array segment

Binary Search

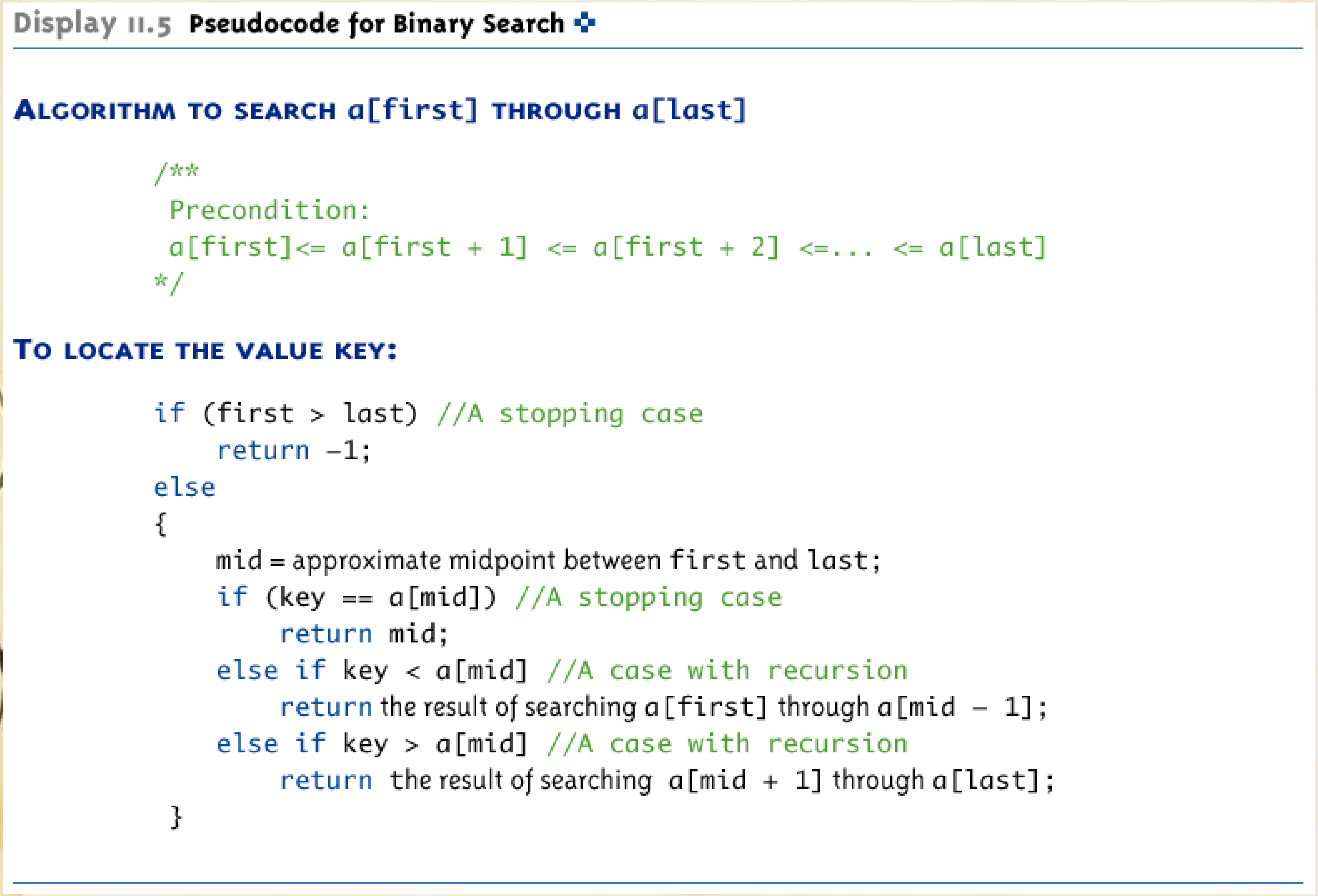
- If the value looked for is larger than the value in the middle of the array or array segment
  - Then the first half of the array or array segment can be ignored
  - This strategy is then applied to the second half of the array or array segment
- If the value looked for is at the middle of the array or array segment, then it has been found
- If the entire array (or array segment) has been searched in this way without finding the value, then it is not in the array

Binary search animation

- searching for number “333” in a sorted list



Pseudocode for Binary



Display 11.5 **Pseudocode for Binary Search**

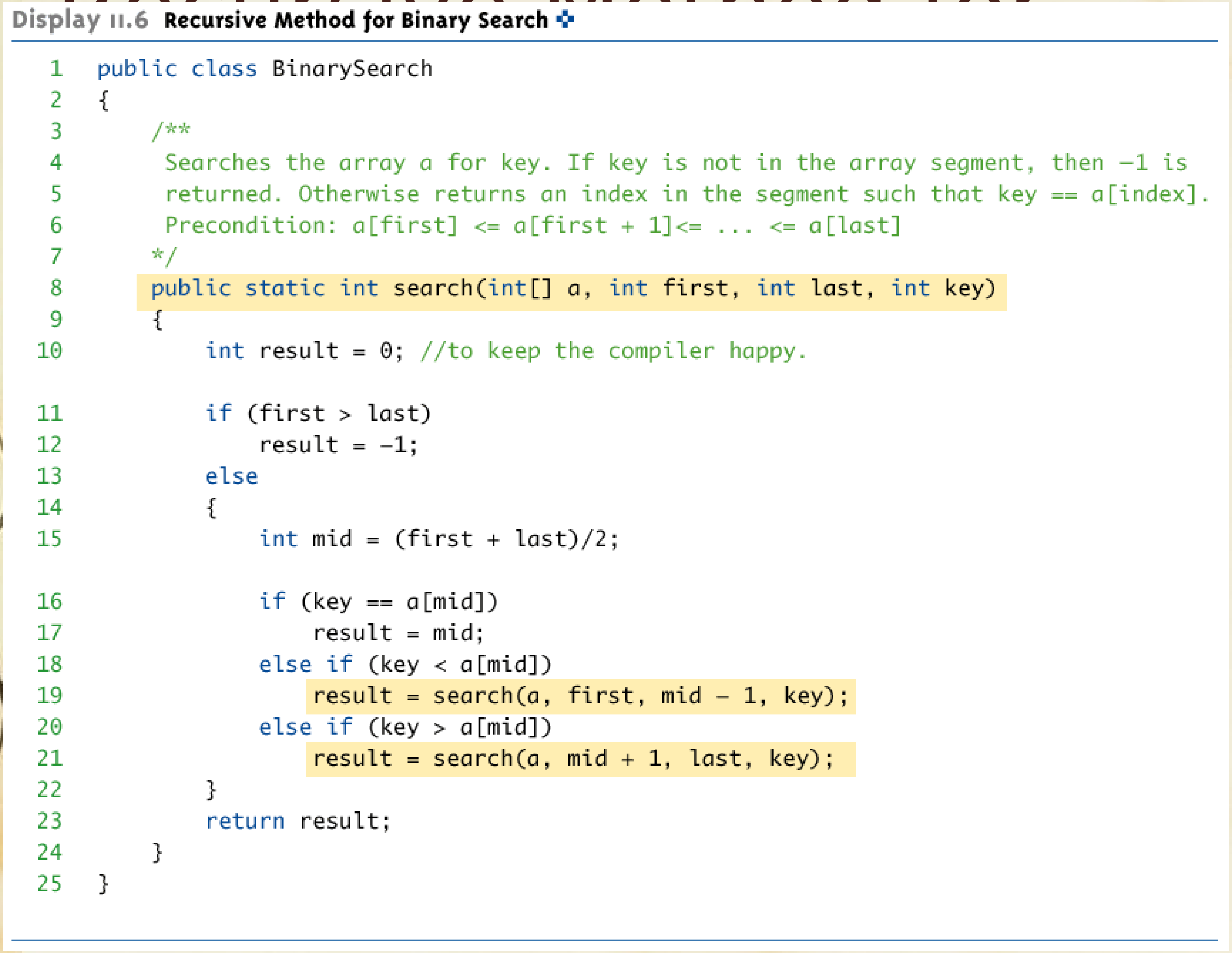
ALGORITHM TO SEARCH a[first] THROUGH a[last]

```
/**
 Precondition:
 a[first]<= a[first + 1] <= a[first + 2] <=... <= a[last]
*/
```

TO LOCATE THE VALUE KEY:

```
if (first > last) //A stopping case
    return -1;
else
{
    mid = approximate midpoint between first and last;
    if (key == a[mid]) //A stopping case
        return mid;
    else if key < a[mid] //A case with recursion
        return the result of searching a[first] through a[mid - 1];
    else if key > a[mid] //A case with recursion
        return the result of searching a[mid + 1] through a[last];
 }
```

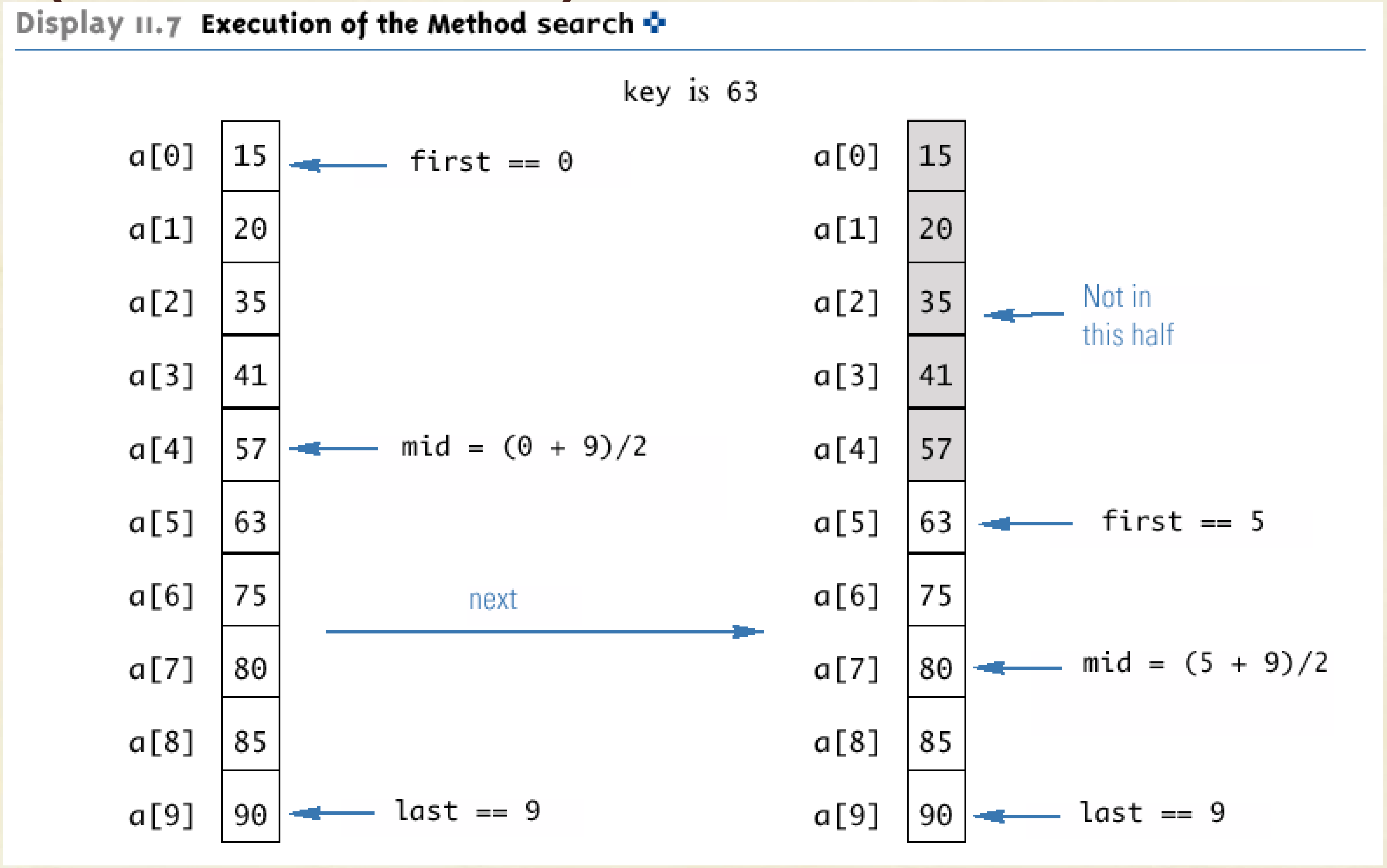
Recursive Method for



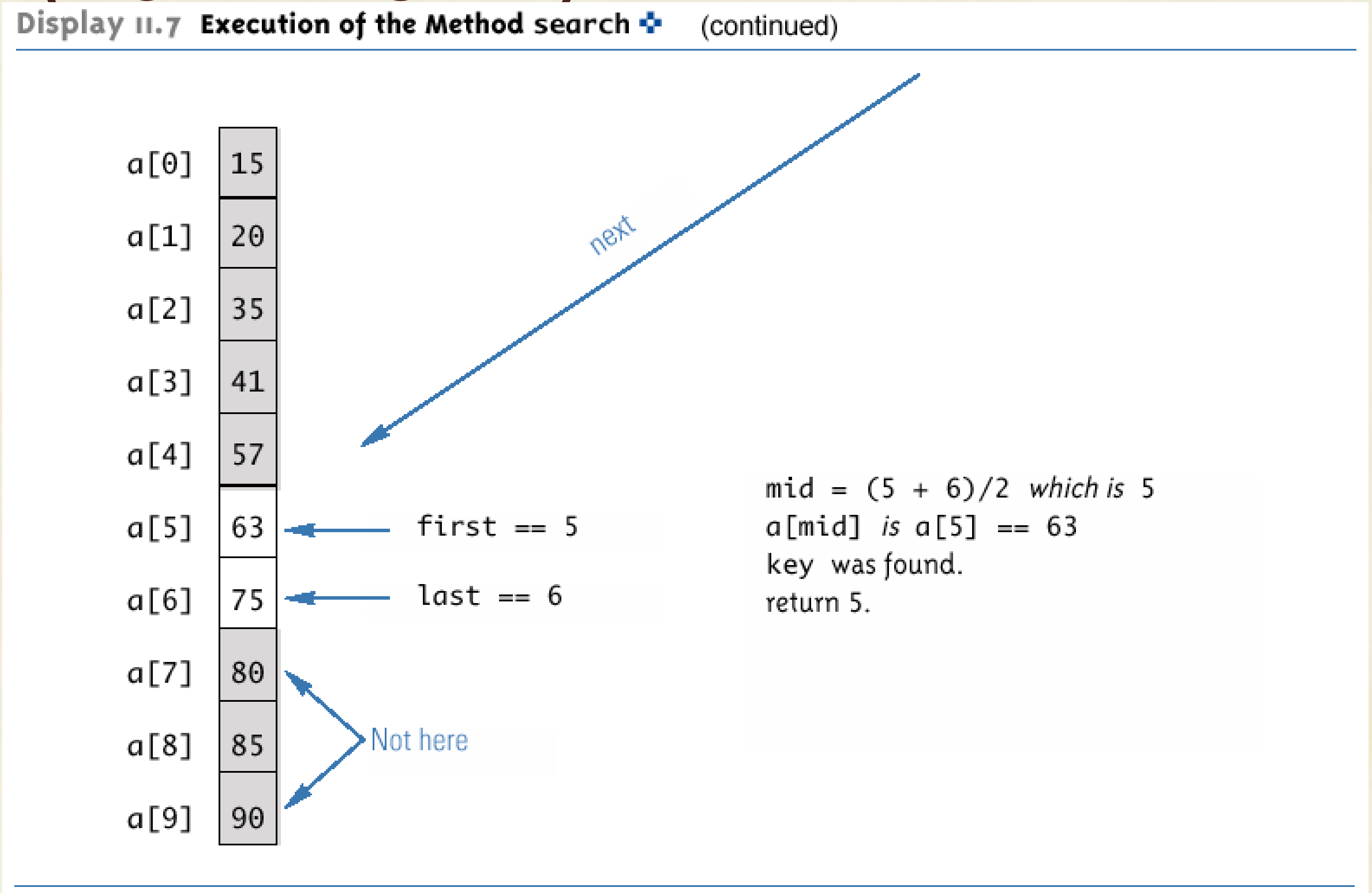
Display 11.6 Recursive Method for Binary Search

```
public class BinarySearch
{
    /**
     Searches the array a for key. If key is not in the array segment, then -1 is
     returned. Otherwise returns an index in the segment such that key == a[index].
     Precondition: a[first] <= a[first + 1]<= ... <= a[last]
    */
    public static int search(int[] a, int first, int last, int key)
    {
        int result = 0; //to keep the compiler happy.

        if (first > last)
            result = -1;
        else
        {
            int mid = (first + last)/2;

            if (key == a[mid])
                result = mid;
            else if (key < a[mid])
                result = search(a, first, mid - 1, key);
            else if (key > a[mid])
                result = search(a, mid + 1, last, key);
        }
        return result;
    }
}
```

Execution of the Method **`search`** (Part 1 of 2)


Display 11.7 Execution of the Method search

Execution of the Method **search** (Part 1 of 2)



Display 11.7 Execution of the Method search (continued)

mid = (5 + 6)/2 *which is* 5
a[mid] *is* a[5] == 63
key was found.
return 5.

Checking the **`search`** Method

1. There is no infinite recursion
   - On each recursive call, the value of **`first`** is increased, or the value of **`last`** is decreased
   - If the chain of recursive calls does not end in some other way, then eventually the method will be called with **`first`** larger than **`last`**

Checking the `search` Method

1. Each stopping case performs the correct action for that case
   - If `first > last`, there are no array elements between `a[first]` and `a[last]`, so `key` is not in this segment of the array, and `result` is correctly set to `-1`
   - If `key == a[mid]`, `result` is correctly set to `mid`

Checking the `search` Method

1. For each of the cases that involve recursion, *if* all recursive calls perform their actions correctly, *then* the entire case performs correctly
   - If `key < a[mid]`, then `key` must be one of the elements `a[first]` through `a[mid-1]`, or it is not in the array
   - The method should then search only those elements, which it does
   - The recursive call is correct, therefore the entire action is correct

Checking the `search` Method

- If `key > a[mid]`, then `key` must be one of the elements `a[mid+1]` through `a[last]`, or it is not in the array
- The method should then search only those elements, which it does
- The recursive call is correct, therefore the entire action is correct

The method `search` passes all three tests:

Therefore, it is a good recursive method definition

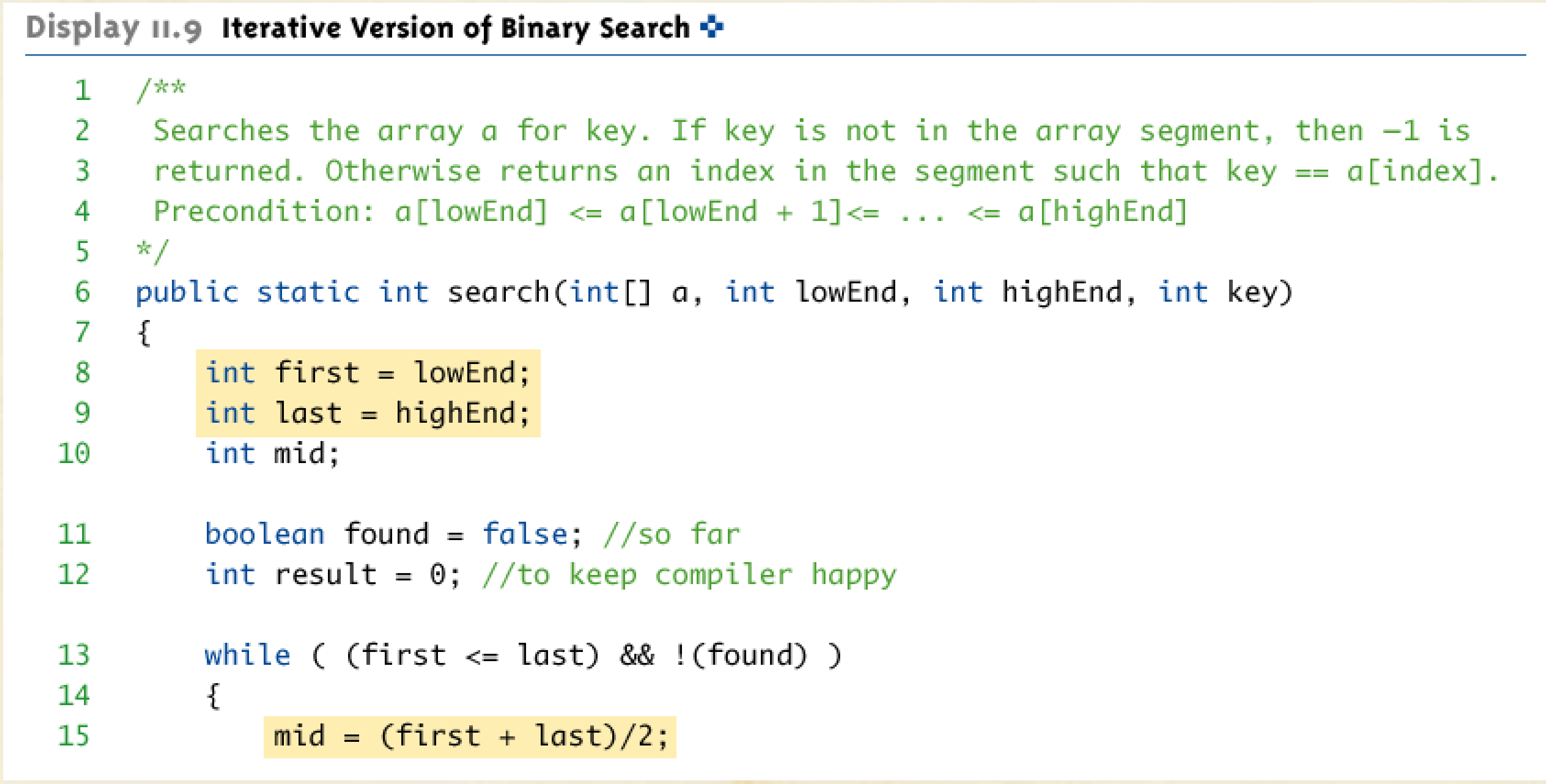
Efficiency of Binary Search

- The binary search algorithm is extremely fast compared to an algorithm that tries all array elements in order
  - About half the array is eliminated from consideration right at the start
  - Then a quarter of the array, then an eighth of the array, and so forth

Efficiency of Binary Search

- Given an array with 1,000 elements, the binary search will only need to compare about 10 array elements to the key value, as compared to an average of 500 for a serial search algorithm
- The binary search algorithm has a worst-case running time that is logarithmic: O(log *n*)
  - Specifically, 1 + $log_2 N$ iterations are needed to return an answer
  - A serial search algorithm is linear: O(*n*)
- If desired, the recursive version of the method `search` can be converted to an iterative version that will run more efficiently

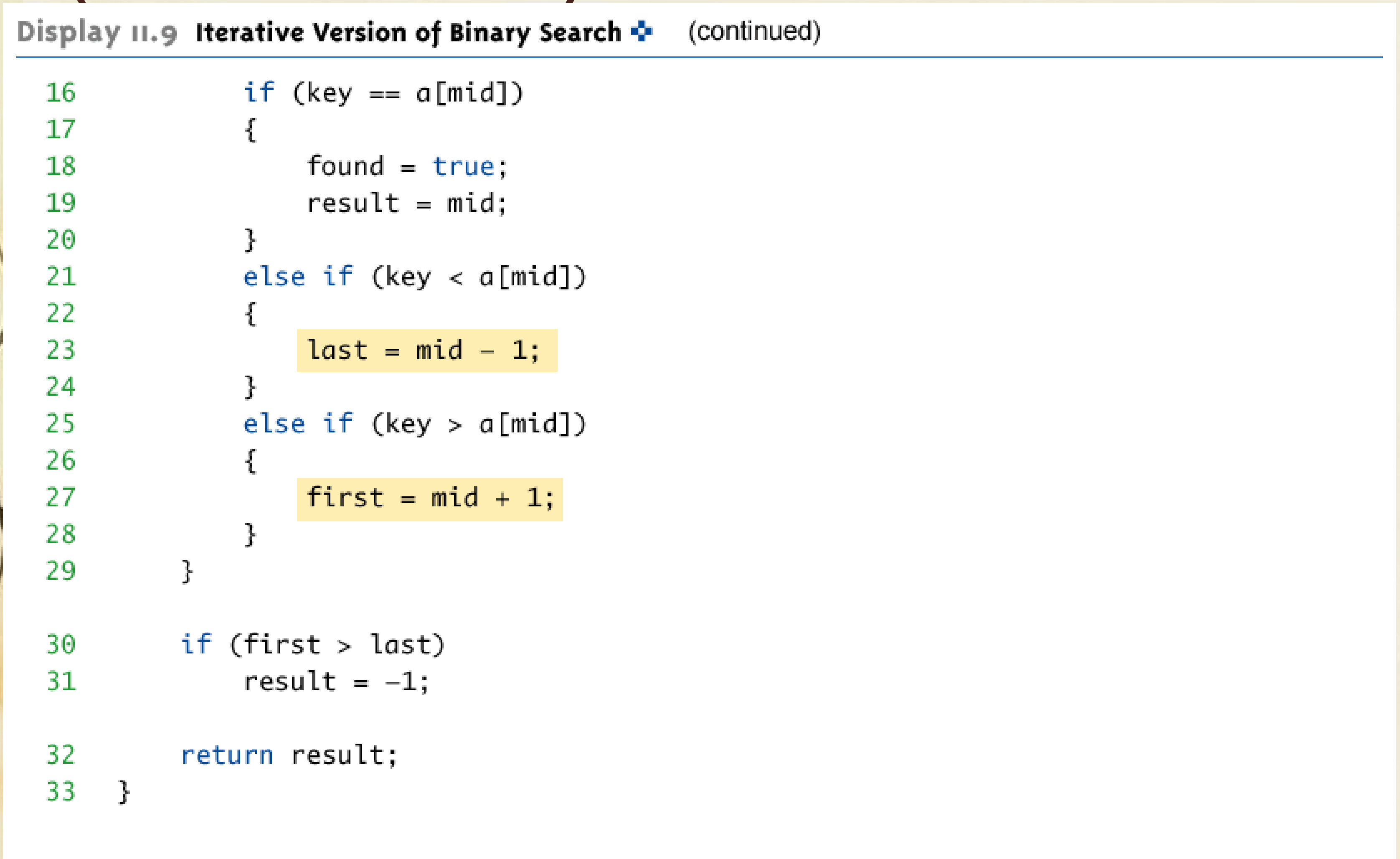
Iterative Version of Binary Search (Part 1 of 2)



Display 11.9 Iterative Version of Binary Search

```
/**
 Searches the array a for key. If key is not in the array segment, then −1 is
 returned. Otherwise returns an index in the segment such that key == a[index].
 Precondition: a[lowEnd] <= a[lowEnd + 1]<= ... <= a[highEnd]
*/
public static int search(int[] a, int lowEnd, int highEnd, int key)
{
    int first = lowEnd;
    int last = highEnd;
    int mid;

    boolean found = false; //so far
    int result = 0; //to keep compiler happy

    while ( (first <= last) && !(found) )
    {
        mid = (first + last)/2;
```

Iterative Version of Binary Search (Part 2 of 2)



Display 11.9 Iterative Version of Binary Search (continued)

```
        if (key == a[mid])
        {
            found = true;
            result = mid;
        }
        else if (key < a[mid])
        {
            last = mid - 1;
        }
        else if (key > a[mid])
        {
            first = mid + 1;
        }
    }

    if (first > last)
        result = -1;

    return result;
}
```

Further exercises

- factorial function:

Further exercises

- Recursive method to produce the nth number in a Fibonacci sequence

The definition of fib is interesting, because it calls itself twice when recursion is used. Consider the effect on program performance of such a function calculating the fibonacci function of a moderate size number.

Recursion - Justification

- there are problems whose solutions are inherently recursive, because they need to keep track of prior state. e.g.:
  - tree traversal
  - divide-and-conquer algorithms such as Quicksort.
- All of these algorithms can be implemented iteratively with the help of a stack, but the need for the stack arguably nullifies the advantages of the iterative solution.
- Also, key concept for functional programming languages, such as Haskell

History of Recursion

- Originally specified in 1958, Lisp is the second-oldest high-level programming language in widespread use today; only Fortran is older.
- recursion a key component of language
- Driven by AI

Break:

- Edsgar Dijkstra?

Edsgar Dijkstra

- The competent programmer is fully aware of the strictly limited size of his own skull; therefore he approaches the programming task in full humility, and among other things he avoids clever tricks like the plague. [9]
- Progress is possible only if we train ourselves to think about programs without thinking of them as pieces of executable code.
- Being abstract is something profoundly different from being vague.

Edsgar Dijkstra

- Nothing is as expensive as making mistakes.
- Program testing can at best show the presence of errors but never their absence.
- ... if 10 years from now, when you are doing something quick and dirty, you suddenly visualize that I am looking over your shoulders and say to yourself, Dijkstra would not have liked this, well that would be enough immortality for me.

Some advanced examples of RECURSIVE procedures

- Quicksort
- Towers of Hanoi
- Trees

QUICKSORT

- A VERY fast RECURSIVE sort
  - $O(n\log_2 n)$ if elements are occur randomly
  - $O(n^2)$ if elements do not occur randomly
- Invented by CAR Hoare
- A simple version is described
  - We are really interested in RECURSION
- An example is done
- Finally - Hints are given on making Quicksort more efficient

Quicksort - Algorithm

- Sort array elements into ascending order
- Choose a pivot element
  - Easiest choice is 1st element of the array
- Compare all elements of array with pivot
  - Elements < pivot go to FRONT of array
  - Elements > pivot go to REAR of array
  - Put pivot in empty slot (it is in correct place)
- Quicksort SMALLER elements
- Quicksort LARGER elements

QUICKSORT

```
void quicksort(int [ ] a, int left, int right)
{  int pos;
   if ( left >= right) return; // terminating
   condition
   pos = rearrange(a, left, right);
   quicksort (a, left, pos-1);
   quicksort (a, pos+1, right);
}
```

Example

	0	1	2	3	4	5	6
	Lef t						Righ t
Initially	6	2	4	3	8	1	7
	2						
	2	**4**					
	2	4	3				
	2	4	3				8
	2	4	3	1			8
	2	4	3	1		7	8
	2	4	3	1	6	7	8
					pos		

Example (cont)

- After rearrangement
  - value 6 is in correct position (element 4)
  - pos = 4
  - quicksort (a, left, pos-1) ➔ (a, 0, 3)
  - quicksort (a, pos+1, right) ➔ (a, 5, 6)

Example (cont)

- quicksort(a, 0, 3) // left to you to do
- quicksort(a, 5, 6) becomes
  - rearrange elements 5 & 6 (values 7   8)
    - (nothing done – values in correct positions)
    - left =5; right = 6; pos = 5
  - quicksort (a, 5, 4)   quicksort(a, left, pos-1)
    - returns immediately as 5 >= 4
    - nothing to sort
  - quicksort (a, 6, 6)   quicksort(a, pos+1, right)
    - returns immediately as 6 >=6
    - nothing to sort

Hints are given on making Quicksort more efficient

- rearrange routine
  - Present routine requires 2 arrays
  - wasteful on space
  - a clever algorithm exists that only requires 1 array
    - not described here because it is complex

Hints are given on making Quicksort more efficient

- Choice of PIVOT element
- We chose the 1st element
  - Easy to implement but a poor choice
- We really want a pivot that splits the array approx equally in half
- Better choice is the median element
  - More difficult to choose (2 approximations)
    - Can choose a random element
    - Can choose the middle element of the 1st, last & middle element

Towers of HANOI

- We have 3 pegs – A B C
- On one peg (A) we have a number of disks
- Each disk is SMALLER than the one it rests on
- Problem
  - Move all disks from peg A to peg B
  - We can only move 1 disk at a time
  - Disks must start & finish resting on a LARGER disk

Towers of Hanoi - Algorithm

- Base Case – if there is 1 disk
  - just move it to directly (ie from fromPeg to toPeg)
- 2 disks
  - move top disk from fromPeg to helpPeg
  - We move bottom disk from fromPeg to toPeg
  - Move remaining disk from helpPeg to toPeg
- 3 disks
- We 1st move 2 disks from fromPeg to helpPeg
  - Smaller problem & we know how to do it recursively
- We move bottom disk from fromPeg to toPeg
- Move 2 disks from helpPeg to toPeg
  - Smaller problem & we know how to do it recursively

Towers of Hanoi - Algorithm

- General case
- We 1st move all but bottom disk from fromPeg to helpPeg
  - Smaller problem & we know how to do it recursively
- We move bottom disk from fromPeg to toPeg
- Move remaining disks from helpPeg to toPeg

RECURSIVE solution

```
Public static void hanoi(int n, Peg
   fromPeg, Peg toPeg, Peg helpPeg)
{ if (n==1) moveDisk ( fromPeg, toPeg);
  else
  { hanoi ( n-1, fromPeg, helpPeg, toPeg);
   moveDisk (fromPeg, toPeg);
   hanoi ( n-1, helpPeg, toPeg, fromPeg)
}}
```

moveDisk routine

```
Public static void moveDisk
              (Peg fromPeg   Peg toPeg)
{
 System.out.println(“Move disk from
   “ + fromPeg + “ to” + toPeg);
}
```

Example 1

hanoi (1, pegA, pegB, pegC)

Move disk from pegA to pegB

Example 2

hanoi (2, pegA, pegB, pegC)

```
Move disk from   pegA   to   pegC
Move disk from   pegA   to   pegB
Move disk from   pegC   to   pegB
```

Example 3

hanoi (3, pegA, pegB, pegC)

Move disk from pegA to pegB
Move disk from pegA to pegC
Move disk from pegB to pegC
Move disk from pegA to pegB
Move disk from pegC to pegA
Move disk from pegC to pegB
Move disk from pegA to pegB

Comments on Towers of Hanoi

1. There is an iterative algorithm but it is very difficult
2. This problem was invented in 1883. See: Mathematical Puzzles & Diversions by Martin Gardner (Penguin 1965)