Number Systems and Logic

UCT Dept of Computer Science
CS115 ~ 2003

Number Representations

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- Numeric info fundamental - encodes data and instructions
- We are used to base/radix ten: decimal (0-9)
- Computers use presence/absence of voltage
  - **binary system**: (0-1) or “off/on”
- General radix *r* number rep:

  $d_p d_{p-1} d_{p-2} \ldots d_2 d_1 d_0 . d_{-1} d_{-2} \ldots d_{-q}$
- Numeric value is: $\Sigma_{i=-q \ldots p}\ d_i r^i$

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- Write: N_r for number N using radix r
- Examples

  Decimal:

  $1041.2_{10} = 1*10^3 + 0*10^2 + 4*10^1 + 1*10^0 + 2*10^{-1}$

  Binary:

  $1001.1_2 = 1*2^3 + 0*2^2 + 0*2^1 + 1*2^0 + 1*2^{-1} = 9.5_{10}$
- n-bit binary number: 0_{10} to $(2^n-1)_{10}$
- Largest 8-bit (unsigned) number: $11111111_2 = 255_{10}$

Binary to Decimal Conversion

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1. quot = number; i = 0;
2. **repeat** until quot = 0:
   1. quot = quot/2;
   2. digit i = remainder;
   3. i++;

- gives digits from least to most signif.

33/2 = 16	rem 1 least sig. digit
16/2 = 8	rem 0
8/2 = 4	rem 0
4/2 = 2	rem 0
2/2 = 1	rem 0
1/2 = 0	rem 1; most sig. digit

Converting fractional numbers

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1. i = 0;
2. **repeat** until N == 1.0 or i == n:
   1. N = FracPart(N); N *= 2;
   2. digit i = IntPart(N); i++

- Eg: 0.125_{10} to binary -> 0.001_2
  0.125*2 = 0.25; IntPart = 0 most significant digit
  0.250*2 = 0.50; IntPart = 0
  0.500*2 = 1.00; IntPart = 1 least significant digit
- Convert int and frac. part separately
- Many numbers cannot be represented accurately:
  0.3_{10} = $[0.0[1001]...]_2$ (bracket repeats, limited by bit size)

Binary Addition

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- Adding binary numbers:
  1+0 = 0+1 = 1; 0 + 0 = 0; 1 + 1 = 0 carry 1
- Possibility of **overflow**
  Add $109_{10}$to 136_{10}:
  01101101 + 10001000 = 11110101 = 245_{10}
  Add 254_{10} to 2_{10}:
  11111110 + 00000010 = [1]00000000 = 256_{10}
- We only have 8 bits to store answer...so it's zero!
- Program can generate an “exception”' to let us know
- Usually number of bits is quite large: eg MIPS R4000 32-bits.

Signed Numbers

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- Can use left-most bit to code sign (0 = pos/1 = neg)
  - Gives symmetric numbers from $-(2^7-1) \ldots 2^7-1$ AND two zeros!!!
  - Addition not straight forward (bad for hardware implementors)
- This is nonsensical and wasteful: can use extra bit pattern
- Try *one's complement:*
  - negative numbers obtained by flipping signs
  - positive numbers unchanged
  - e.g. -5 = complement(00000101) = 11111010
- Left-most bit still indicates sign

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- Now easy to subtract: complement number and add:

  e.g. 5 - 6

  = 00000101 + complement(00000110)

  = 00000101 + 11111001

  = 11111110

  = complement(00000001) (-1)
- A carry is added into right-most bit
- Can still overflow: can check sign bits
  - Only numbers with same sign can overflow
  - Check: if input sign != output sign then overflow

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- Evaluate 10–7 using 8-bit one's complement arithmetic:

  10 - 7

  = 00001010 + complement(00000111)

  = 00001010 + 11111000

  = 00000010 carry 1

  = 00000010 + 00000001

  = 00000011 = 3_{10}

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- Still have two zeros $\Rightarrow$ *two's complement*
  - Complement then add 1
  - Our number range now asymmetric: $-2^7 \ldots 2^7-1$
  - Used extra zero bit pattern
- Now when we add, discard carry

  10 - 7

  = 00001010 + 2complement(00000111)

  = 00001010 + 11111001

  = 00000011 carry 1 (discard)

  = 3
- Same overflow test can be used

Binary Coded Decimal

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- Can use Binary Coded Decimal (BCD) to represent integers:
  - map 4 bits per digit (from 0000)
- Wasteful: only 10 bit patterns reqd; 6 wasted. Binary more compact code e.g.
  - $256_{10} = 100000000_2 = 0010\ 0101\ 0110_{BCD}$
  - so 9 vs 12 bits in this case
- Not practical; complicates hardware implementation
  - How do you add/subtract, deal with carries etc?

Octal and Hexadecimal

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- Base 8 (octal) and base 16 (Hexadecimal) are sometimes used (powers of 2)
- Octal (0NNN...N) uses digits 0-7
- Hex (0xNNN...N) uses “digits” 0-9,A-F
- Examples: $17_{10} = 10001_2 = 21_8 = 11_{16}$
- Conversion as for decimal to binary:
  - divide/multiply by 8 or 16 instead
- Binary to octal or hexadecimal
  - group bits into 3 (octal) or 4 (hex) from LS bit
  - pad with leading zeros if reqd

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- 0100011011010111_2
  = (000) (100) (011) (011) (010) (111)
  = 43327_8
  = (0100) (0110) (1101) (0111)
  = $46D7_{16}$
- Note padding at front of number

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- To convert from hex/octal to binary: reverse procedure
  $FF_{16} = (1111)(1111)_2$
  $377_8 = (011)(111)(111)_2$
- NOTE: for fractional conversion, move from left to right and pad at right end:
  $0.11001101011_2 = 0.\ (110)\ (011)\ (010)\ (110)$
  $= 0.6326_8$
  $0.11_2 = 0.(110)_2 = 0.6_8$
- Convert fractional/integer part separately
- When converting to hex/oct may be easier to conv. to bin first

Floating point Numbers

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- Fixed point numbers have very limited range (determined by bit length)
- 32-bit value can hold integers from -2^{31} to $2^{31}-1$ or smaller range of fixed point fractional values
- Solution: use *floating point* (scientific notation)

  Thus 0.0000000000000976 $\Rightarrow$ $9.76*10^{-14}$
- Consists of two parts: **mantissa** & **exponent**
  - **Mantissa:** the number multiplying the base
  - **Exponent:** the power
- The **significand** is the part of the mantissa after the decimal point

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- Range of numbers is very large, but accuracy is limited by significand
- So, for 8 digits of precision,

  $976375297321 = 9.7637529*10^{11}$,

  and we loose accuracy (**truncation error**)
- can normalise any floating point number:

  $34.34*10^{12} = 3.434*10^{13}$
- Shift point until only one non-zero digit is to left
  - add 1 to exponent for each left shift
  - subtract 1 for each right shift

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- Can use notation for binary: use base of 2
  $0.11001*2^{-3} = 1.11001*2^{-4} = 1.11001 * 2^{11111100}$ (2's complement exponent)
- For binary FP numbers, normalise to:
  $1.xxx\ldots xxx*2^{yy\ldots yy}$
- Problems with FP:
  - Many different floating point formats; problems exchanging data
  - FP arithmetic not associative: x + (y + z) != (x + y) + z
- IEEE 754 format introduced: single (32-bit) and double (64-bit) formats; standard!

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- Also extended precision - 80 bits (long double).
- Single precision number represented internally as
  - sign bit
  - followed by exponent (8-bits)
  - then the fractional part of normalised number (23 bits)
- The leading 1 is implied; not stored
- Double precision
  - has 11-bit exponent and
  - 52-bit significand
- Single precision range: $2*10^{-38}$ to $2*10^{38}$
- Double range: $2*10^{-308}$ to $2*10^{308}$

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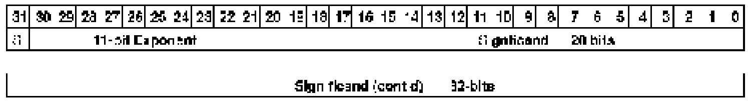
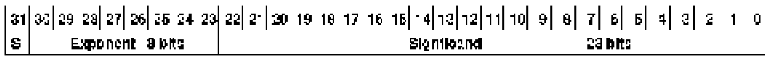


IEEE 754 - Double (64-bit) floating point format



IEEE 754 - Single (32-bit) floating point format

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- The exponent is “biased‘”: no explicit negative number
- Single precision: 127, Double precision 1023
- So, for single prec:

  exponent of 255 is same as 255-127 = 128, and 0 is 0 - 127 = -127 (can't be symmetric, because of zero)
- Most positive exponent: 111...11, most negative: 00....000
- Makes some hardware/logic easier for exponents (easy sorting/compare)
- numeric value of stored IEEE FP is actually:

  $(-1)^S * (1 + \text{significand}) * 2^{\text{exponent - bias}}$

Example: -0.75 to IEEE754 Single

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- Sign is negative: so S = 1
- Binary fraction:
  - 0.75*2 = 1.5 (IntPart = 1)
  - 0.50*2 = 1.0 (IntPart = 1), so $0.75_{10} = 0.11_2$
- Normalise: $0.11*2^0 = 1.1*2^{-1}$
- Exponent: -1, add bias(127) = 126 = 01111110;
- Answer: 1 01111110 100…000000000

  s 8 bits 23 bits

What is the value of this FP num?

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1 10000001 10010000000000000000000

1. Negative number (s=1)
2. Biased exponent: 10000001 = 128+1 = 129
   1. Unbiased exponent = 129-127 = 2
3. Significand: 0.1001 = 0.5+0.0625 = 0.5625
4. Value = $-1 * (1 + 0.5625)*2^2 = -6.25_{10}$

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- IEEE 754 has special codes for zero, error conditions (0/0 etc)
- **Zero**: exponent and significand are zero
- **Infinity**: exp = 1111...1111, significand = 0
- **NaN** (not a number): 0/0; exponent = 1111...1111, significand != 0
- Underflow/overflow conditions:

Range of single prec. float

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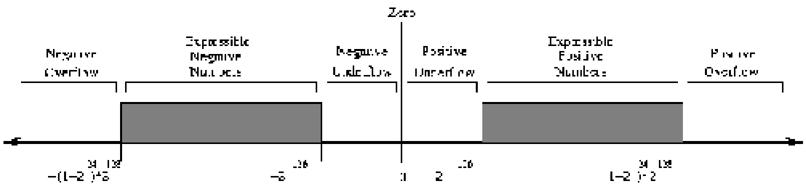

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- Addition/Subtraction: normalise, match to larger exponent then add, normalise
- Error conditions:
  - **Exponent Overflow** Exponent bigger than max permissable size; may be set to “infinity”'
  - **Exponent Underflow** Neg exponent, smaller than minimum size; may be set to zero
  - **Significand Underflow** Alignment may causes loss of significant digits
  - **Significand Overflow** Addition may cause carry overflow; realign significands

Character Representations

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- Characters represented using “character set”
- Examples:
  - ASCII (8-bit)
  - Unicode (16-bit)
  - EBCDIC (9-bit)
- ASCII - American Standard Code for Information Interchange
- Widely used; 7-bits used for std characters etc.; extra for parity or foreign language

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- ASCII codes for roman alphabet, numbers, keyboard symbols and basic network control
- Parity-bit allows error check (crude) cf. Hamming codes
- Unicode quite new: subsumes ASCII, extensible, supported by Java
- Handles many languages, not just roman alphabet and basic symbols

Bit/Byte Ordering

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- Endianess: ordering of bits or bytes in computer
  - Big Endian: bytes ordered from MSB to LSB
  - Little Endian: bytes ordered from LSB to MSB
- Example: how is Hex A3 FC 6D E5 (32-bit) represented?
  - Big Endian: A3FC6DE5 (lowest byte address stores MSB)
  - Little Endian: E56DFCA3 (lowest byte address stores LSB)
- Problems with multi-byte data: floats, ints etc
- MIPS Big Endian, Intel x86 Little Endian
- Bit ordering issues as well: endian on MSb/LSb
- Can check using bitwise operators...

Boolean Algebra & Logic

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- Modern computing devices are digital rather than analog
  - Use two discrete states to represent all entities: 0 and 1
  - Call these two logical states **TRUE** and **FALSE**
- All operations will be on such values, and can only yield such values
- George Boole formalised such a logic algebra: “Boolean Algebra”
- Modern digital circuits are designed and optimised using this theory
- We implement “functions” (such as add, compare, etc.) in hardware, using corresponding Boolean expressions

Boolean Operators

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- There are 3 basic logic operators

Operator	Usage	Notation
AND	A AND B	A.B
OR	A OR B	A+B
NOT	NOT A	$\overline{A}$

- A and B can only be TRUE or FALSE
- TRUE represented by 1; FALSE by 0

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- To show the value of each operator (or combinations thereof) we can use a Truth Table
  - AND is TRUE only if both args are TRUE
  - OR is TRUE if either is TRUE
  - NOT is a unary operator: inverts truth value

A	B	F=A.B	F=A+B	F = $\overline{A}$	F=$\overline{B}$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	0	1	0	1
1	1	1	1	0	0

NAND, NOR and XOR

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a) NAND is FALSE only both args are TRUE [NOT (A AND B)]

b) NOR is TRUE only if both args are FALSE [NOT (A OR B)]

c) XOR is TRUE is either input is TRUE, but not both

A	B	F=$\overline{A.B}$	F=$\overline{A+B}$	F=A⊕B
0	0	1	1	0
0	1	1	0	1
1	0	1	0	1
1	1	0	0	0

Logic Gates

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- These operators have symbolic representations: “logic gates”
- Building blocks for all computer circuits
- Can specify arbitrary F using truth table; and derive Boolean expression

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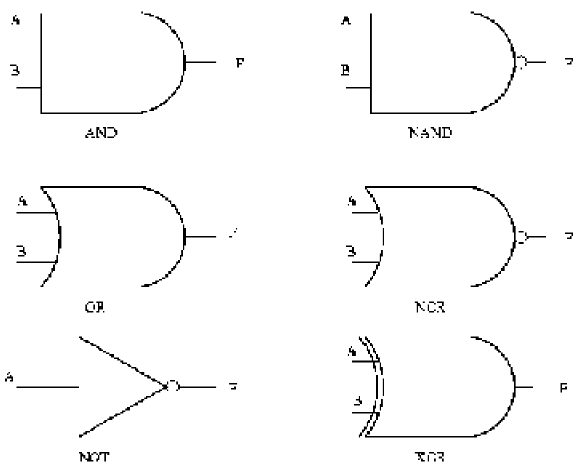

Finding a Boolean Representation

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- F = F(A,B,C); F called "output variable"
- Find F values which are TRUE:

  So, if A=0, B=1, C=0, then F = 1.
  So, $F_1 = \overline{A}.B.\overline{C}$
  That is, we know our output is
  TRUE for this expression (from the table).
  Also have $F_2 = \overline{A}.B.C$ & $F_3 = A.B.\overline{C}$
  F TRUE if F_1 TRUE or F_2 TRUE or F_3 TRUE
  $\Rightarrow F = F_1 + F_2 + F_3$

Cases for F FALSE follows from F TRUE

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

Algebraic Identities

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- *Commutative*: A.B = B.A and A+B = B+A
- *Distributive*:
  A.(B+C) = (A.B) + (A.C)
  A+(B.C) = (A+B).(A+C)
- *Identity Elements*: 1.A = A and 0 + A = A
- *Inverse*: $A.\overline{A} = 0$ and $A + \overline{A} = 1$
- *Associative*:
  A.(B.C) = (A.B).C and A+(B+C) = (A+B)+C
- *DeMorgan's Laws*:
  $\overline{A.B} = \overline{A} + \overline{B}$ and
  $\overline{A+B} = \overline{A}.\overline{B}$